

Regularity principle in sequence spaces and applications

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Abstract

We prove a nonlinear regularity principle in sequence spaces which produces universal estimates for special series defined therein. Some consequences are obtained and, in particular, we establish new inclusion theorems for multiple summing operators. Of independent interest, we settle all Grothendieck's type (ℓ_1, ℓ_2) theorems for multilinear operators. We further employ the new regularity principle to solve the classification problem concerning all pairs of admissible exponents in the anisotropic Hardy–Littlewood inequality.

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1 Introduction

Regularity arguments are fundamental tools in the analysis of a variety of problems as they often pave the way to important discoveries in the realm of mathematics and its applications. Regularity results may appear in many different configurations, sometimes quite explicitly as in the theory of diffusive PDEs, sometimes in a more subtle form, and in this article we are interested in the following universality problem, which drifts a hidden regularity principle in it:

Problem 1. Let $p \geq 1$ be a real number, X, Y, W_1, W_2 be non-void sets, Z_1, Z_2, Z_3 be normed spaces and $f: X \times Y \rightarrow Z_1$, $g: X \times W_1 \rightarrow Z_2$, $h: Y \times W_2 \rightarrow Z_3$ be particular maps. Assume there is a constant $C > 0$ such that

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \|f(x_i, y_j)\|^p \leq C \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} \|g(x_i, w)\|^p \right) \cdot \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} \|h(y_j, w)\|^p \right), \quad (1)$$

for all $x_i \in X, y_j \in Y$ and $m_1, m_2 \in \mathbb{N}$. Are there (universal) positive constants $\epsilon \sim \delta$, and $\tilde{C}_{\delta, \epsilon}$ such that

$$\left(\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \|f(x_i, y_j)\|^{p+\delta} \right)^{\frac{1}{p+\delta}} \leq \tilde{C}_{\delta, \epsilon} \cdot \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} \|g(x_i, w)\|^{p+\epsilon} \right)^{\frac{1}{p+\epsilon}} \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} \|h(y_j, w)\|^{p+\epsilon} \right)^{\frac{1}{p+\epsilon}}, \quad (2)$$

for all $x_i \in X, y_j \in Y$ and $m_1, m_2 \in \mathbb{N}$?

It turns out that many classical questions in mathematical analysis, permeating several different fields, can be framed into the formalism of the universality Problem 1. A key observation is that the existence of a leeway, $\epsilon > 0$, of an increment $\delta > 0$, and of a corresponding bound $\tilde{C}_{\delta, \epsilon} > 0$ bears a regularity principle for the orderly problem which often reveals important aspects of the theory underneath.

In this work, under appropriate assumptions, we solve the universality problem in a very general setting. This is a flexible, effective tool and we apply it in the investigation of two central problems in mathematical analysis, namely inclusion type theorems for summing operators and the solution of the classification problem in sharp anisotropic Hardy–Littlewood inequality.

The theory of absolutely summing operators plays an important role in the study of Banach Spaces and Operator Theory, with deep inroads in other areas of Analysis. Grothendieck’s inequality, described by Grothendieck as “the fundamental theorem in the metric theory of tensor products” is one of the cornerstones of the theory of absolutely summing operators, and a fundamental general result in Mathematics [9, 13, 25]. For linear operators, p -summability implies q -summability whenever $1 \leq p \leq q$. More generally, if $1 \leq p_j \leq q_j$, $j = 1, 2$, every absolutely $(p_1; p_2)$ -summing operators is absolutely $(q_1; q_2)$ -summing whenever

$$\frac{1}{p_2} - \frac{1}{p_1} \leq \frac{1}{q_2} - \frac{1}{q_1}.$$

Results of this sort are usually called “inclusion results”. In the multilinear setting inclusion results are more intriguing. For instance, every multiple p -summing multilinear operator is multiple q -summing whenever $1 \leq p \leq q \leq 2$, but this is not valid beyond the threshold 2. Our first application of the regularity principle provides new inclusion theorems for multiple summing operators overtaking the barrier 2. Our proof is based on delicate inclusion properties that follow as consequence of the general regularity principle we will establish.

The second featured application we carry on pertains to the theory of anisotropic Hardy-Littlewood inequality. Given numbers $p, q \in [2, \infty]$ and a pair of exponents (a, b) , one is interested in the existence of a universal constant $C_{p,q,a,b} \geq 1$ such that

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \leq C_{p,q,a,b} \cdot \|T\|, \quad (3)$$

for all bilinear operators $T: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$ and all positive integers n ; here and henceforth \mathbb{K} denotes the field of real or complex scalars. Questions of this sort are essential in many areas of mathematical analysis and dates back, at least, to the works of Toeplitz [29] and Riesz [27]. Hardy and Littlewood, in [14], establish the existence of particular anisotropic exponents for which (3) holds and since then a key issue in the theory has been to investigate the optimal range of anisotropic exponents. As an application of the regularity principle, we obtain a complete classification of all pairs of anisotropic exponents (a, b) for which estimate (3) holds, providing henceforth a definitive solution to the problem. We show that (3) is verified if, and only if, the pair of anisotropic exponents (a, b) lies in $[\frac{q}{q-1}, \infty) \times [\frac{pq}{pq-p-q}, \infty)$ and verifies

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right).$$

In the case (3) fails to hold we obtain the precise dimension blow-up rate. We further comment on generalizations of such results to the multilinear setting.

The paper is organized as follows. In Section 2 we introduce and prove the regularity principle — the *tour of force* of this work. In Section 3, we explore the regularity principle as to establish new inclusion properties for multiple summing operators. In Section 4 we prove an all-embracing Grothendieck's type (ℓ_1, ℓ_2) theorem. Section 5 is devoted to the solution of the anisotropic Hardy–Littlewood inequality problem for bilinear forms. We show that an application of the regularity principle classifies all the admissible anisotropic exponents for which Hardy–Littlewood inequality is valid. In Section 5 we determine the exact blow-up rate for non-admissible Hardy–Littlewood exponents, as dimension goes to infinity. In the final section we discuss some new insights concerning Hardy–Littlewood inequality in the multilinear setting, which may pave the way to further investigations in the theory.

2 The Regularity Principle

In this Section we will establish a nonlinear regularity principle which greatly expands the investigation initiated in [20] concerning inclusion properties for sums in one index. Regularity results for summability in multiple indexes, objective of our current study, are rather more challenging and involve a number of new technical difficulties. Accordingly, it is indeed a rather more powerful analytical tool and we shall explore its full strength in the upcoming sections.

Let Z_1 , V and W_1 , W_2 be arbitrary non-void sets and Z_2 be a vector space. For $t = 1, 2$, let

$$R_t : Z_t \times W_t \longrightarrow [0, \infty), \text{ and} \\ S : Z_1 \times Z_2 \times V \longrightarrow [0, \infty)$$

be arbitrary mappings satisfying

$$R_2(\lambda z, w) = \lambda R_2(z, w), \\ S(z_1, \lambda z_2, v) = \lambda S(z_1, z_2, v)$$

for all real scalars $\lambda \geq 0$. In addition, all along the paper we adopt the convention $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$, and throughout this Section we always work in the range $p_1 \geq 1$, and assume

$$\left(\sup_{w \in W_t} \sum_{j=1}^{m_t} R_t(z_{t,j}, w)^{p_1} \right)^{\frac{1}{p_1}} < \infty, \quad t = 1, 2. \quad (4)$$

Note these are rather general, weak hypotheses on the governing maps S, R_1, R_2 ; in particular no continuity conditions are imposed.

Theorem 2.1 (Regularity Principle). *Let $1 \leq p_1 \leq p_2 < 2p_1$ and assume*

$$\left(\sup_{v \in V} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{p_1} \right)^{\frac{1}{p_1}} \leq C \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_1} \right)^{\frac{1}{p_1}} \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}},$$

for all $z_{1,i} \in Z_1, z_{2,j} \in Z_2$, all $i = 1, \dots, m_1$ and $j = 1, \dots, m_2$ and $m_1, m_2 \in \mathbb{N}$. Then

$$\left(\sup_{v \in V} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{\frac{p_1 p_2}{2p_1 - p_2}} \right)^{\frac{2p_1 - p_2}{p_1 p_2}} \leq C \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}} \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_2} \right)^{\frac{1}{p_2}},$$

for all $z_{1,i} \in Z_1, z_{2,j} \in Z_2$, all $i = 1, \dots, m_1$ and $j = 1, \dots, m_2$ and $m_1, m_2 \in \mathbb{N}$.

Proof. Consider $(z_{1,i})_{i=1}^{m_1}$ in Z_1 and $(z_{2,j})_{j=1}^{m_2}$ in Z_2 and define, for all $z \in Z_1$ and $v \in V$,

$$S_1(z, v) = \left(\sum_{j=1}^{m_2} S(z, z_{2,j}, v)^{p_1} \right)^{1/p_1}.$$

We estimate

$$\begin{aligned}
\sup_{v \in V} \sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_1} &= \sup_{v \in V} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{p_1} \\
&\leq C^{p_1} \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_1} \right) \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1} \right) \\
&= C_1 \sup_{w \in W_1} \sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_1}
\end{aligned}$$

with

$$C_1 = C^{p_1} \sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1}.$$

We can consider a new sequence in Z_1 where each term is repeated with a prescribed frequency. Let η_i be the number of times each $z_{1,i}$ appears respectively. We have

$$\sum_{i=1}^{m_1} \eta_i S_1(z_{1,i}, v)^{p_1} \leq C_1 \sup_{w \in W_1} \sum_{i=1}^{m_1} \eta_i R_1(z_{1,i}, w)^{p_1}, \quad (5)$$

for all $z_{1,1}, \dots, z_{1,m_1} \in Z_1$ and all $v \in V$. Now, passing from integers to rationals by “cleaning” denominators and from rationals to real numbers using density, we conclude that (5) holds for positive real numbers $\eta_1, \dots, \eta_{m_1}$. Define

$$\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}.$$

For each $i = 1, \dots, m_1$, consider the map $\lambda_i: V \rightarrow [0, \infty)$ by

$$\lambda_i(v) := S_1(z_{1,i}, v)^{\frac{p_2}{p}}.$$

Hence, we readily have

$$\begin{aligned}
\lambda_i(v)^{p_1} S_1(z_{1,i}, v)^{p_1} &= S_1(z_{1,i}, v)^{\frac{p_1 p_2}{p}} S_1(z_{1,i}, v)^{p_1} \\
&= S_1(z_{1,i}, v)^{p_2}.
\end{aligned}$$

Recalling that (5) is valid for arbitrary positive real numbers η_i , we get, for $\eta_i = \lambda_i(v)^{p_1}$,

$$\begin{aligned}
\sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_2} &= \sum_{i=1}^{m_1} \lambda_i(v)^{p_1} S_1(z_{1,i}, v)^{p_1} \\
&\leq C_1 \sup_{w \in W_1} \sum_{i=1}^{m_1} \lambda_i(v)^{p_1} R_1(z_{1,i}, w)^{p_1},
\end{aligned}$$

for every $v \in V$. Also, taking into account the relation

$$\frac{1}{(p/p_1)} + \frac{1}{(p_2/p_1)} = 1,$$

and Hölder's inequality we obtain

$$\begin{aligned}
\sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_2} &\leq C_1 \sup_{w \in W_1} \sum_{i=1}^{m_1} \lambda_i(v)^{p_1} R_1(z_{1,i}, w)^{p_1} \\
&\leq C_1 \sup_{w \in W_1} \left[\left(\sum_{i=1}^{m_1} \lambda_i(v)^{\frac{p_1 p_2}{p_1}} \right)^{\frac{p_1}{p_2}} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_1 \frac{p_2}{p_1}} \right)^{\frac{p_1}{p_2}} \right] \\
&= C_1 \left(\sum_{i=1}^{m_1} \lambda_i(v)^{p_1} \right)^{\frac{p_1}{p_2}} \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{p_1}{p_2}} \\
&= C_1 \left(\sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_2} \right)^{\frac{p_1}{p_2}} \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{p_1}{p_2}}.
\end{aligned}$$

Therefore

$$\left(\sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_2} \right)^{1 - \frac{p_1}{p_2}} \leq C_1 \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{p_1}{p_2}}$$

for every $v \in V$, and we can finally conclude that

$$\left(\sup_{v \in V} \sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_2} \right)^{\frac{p_1}{p_2}} \leq C_1 \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{p_1}{p_2}}.$$

Hence

$$\begin{aligned}
\left(\sup_{v \in V} \sum_{i=1}^{m_1} S_1(z_{1,i}, v)^{p_2} \right)^{\frac{1}{p_2}} &\leq C_1^{1/p_1} \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}} \\
&= C \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}} \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}}.
\end{aligned}$$

Recalling the definition of S_1 we have

$$\begin{aligned}
&\left(\sup_{v \in V} \sum_{i=1}^{m_1} \left(\sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{p_1} \right)^{p_2/p_1} \right)^{\frac{1}{p_2}} \\
&\leq C \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}} \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}}.
\end{aligned}$$

Since $p_1 \leq p_2$ we have

$$\begin{aligned}
&\left(\sup_{v \in V} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{p_2} \right)^{\frac{1}{p_2}} \\
&\leq C \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}} \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}}.
\end{aligned}$$

Now we look at the above inequality as

$$\left(\sup_{v \in V} \sum_{j=1}^{m_2} S_2(z_{2,j}, v)^{p_2} \right)^{\frac{1}{p_2}} \leq C_2 \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}}$$

with

$$S_2(z, v) = \left(\sum_{i=1}^{m_1} S(z_{1,i}, z, v)^{p_2} \right)^{\frac{1}{p_2}},$$

$$C_2 = C \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}}$$

for every $z \in Z_2$ and $v \in V$. Since

$$S_2(\lambda z, v) = \lambda S_2(z, v),$$

$$R_2(\lambda z, w) = \lambda R_2(z, w),$$

for all non negative scalars λ , we can use a somewhat similar argument. Recall that

$$\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}$$

and note that

$$\frac{1}{p} = \frac{1}{p_2} - \frac{1}{q},$$

with

$$q = \frac{p_1 p_2}{2p_1 - p_2}.$$

For each $j = 1, \dots, m_2$, consider the map $\vartheta_j: V \rightarrow [0, \infty)$ given by

$$\vartheta_j(v) := S_2(z_{2,j}, v)^{\frac{q}{p}}.$$

We find

$$\begin{aligned} \vartheta_j(v)^{p_2} S_2(z_{2,j}, v)^{p_2} &= S_2(z_{2,j}, v)^{\frac{p_2 q}{p}} S_2(z_{2,j}, v)^{p_2} \\ &= S_2(z_{2,j}, v)^q. \end{aligned}$$

Thus,

$$\begin{aligned}
\left(\sum_{j=1}^{m_2} S_2(z_{2,j}, v)^q \right)^{\frac{1}{p_2}} &= \left(\sum_{j=1}^{m_2} S_2(\vartheta_j(v)z_{2,j}, v)^{p_2} \right)^{\frac{1}{p_2}} \\
&\leq C_2 \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} R_2(\vartheta_j(v)z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}} \\
&= C_2 \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} \vartheta_j(v)^{p_1} R_2(z_{2,j}, w)^{p_1} \right)^{\frac{1}{p_1}} \\
&\leq C_2 \sup_{w \in W_2} \left[\left(\sum_{j=1}^{m_2} \vartheta_j(v)^{p_1 \cdot \frac{p}{p_1}} \right)^{\frac{p_1}{p}} \left(\sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_1 \cdot \frac{p_2}{p_1}} \right)^{\frac{p_1}{p_2}} \right]^{\frac{1}{p_1}} \\
&= C_2 \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} \vartheta_j(v)^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_2} \right)^{\frac{1}{p_2}}.
\end{aligned}$$

Hence

$$\left(\sum_{j=1}^{m_2} S_2(z_{2,j}, v)^q \right)^{\frac{1}{p_2} - \frac{1}{p}} \leq C_2 \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_2} \right)^{\frac{1}{p_2}},$$

i.e.,

$$\left(\sum_{j=1}^{m_2} S_2(z_{2,j}, v)^q \right)^{\frac{1}{q}} \leq C \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}} \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_2} \right)^{\frac{1}{p_2}}.$$

We thus have

$$\left(\sum_{j=1}^{m_2} \left(\sum_{i=1}^{m_1} S(z_{1,i}, z_{2,j}, v)^{p_2} \right)^{\frac{1}{p_2} \cdot q} \right)^{\frac{1}{q}} \leq C \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}} \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_2} \right)^{\frac{1}{p_2}},$$

and since $p_2 \leq q$, we have

$$\left(\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^q \right)^{\frac{1}{q}} \leq C \sup_{w \in W_1} \left(\sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{p_2} \right)^{\frac{1}{p_2}} \sup_{w \in W_2} \left(\sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{p_2} \right)^{\frac{1}{p_2}},$$

which finally completes the proof of Theorem 2.1. \square

The reasoning developed in the proof of Theorem 2.1 can also be employed as to produce a more general result. Following the previous set-up, let $k \geq 2$ and Z_1, V and W_1, \dots, W_k be arbitrary non-void sets and Z_2, \dots, Z_k be vector spaces. For $t = 1, \dots, k$, let

$$\begin{aligned}
R_t &: Z_t \times W_t \longrightarrow [0, \infty), \text{ and} \\
S &: Z_1 \times \dots \times Z_k \times V \longrightarrow [0, \infty)
\end{aligned}$$

be arbitrary mappings satisfying

$$R_t(\lambda z, w) = \lambda R_t(z, w),$$

$$S(z_1, \dots, z_{j-1}, \lambda z_j, z_{j+1}, \dots, z_k, v) = \lambda S(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_k, v)$$

for all scalars $\lambda \geq 0$ and all $j, t = 2, \dots, k$.

Theorem 2.2 (Regularity Principle for k -variables). *Let $1 \leq p_1 \leq p_2 < \frac{kp_1}{k-1}$, and assume*

$$\left(\sup_{v \in V} \sum_{j_1=1}^{m_1} \cdots \sum_{j_k=1}^{m_k} S(z_{1,j_1}, \dots, z_{k,j_k}, v)^{p_1} \right)^{\frac{1}{p_1}} \leq C \prod_{t=1}^k \left(\sup_{w \in W_t} \sum_{j=1}^{m_t} R_t(z_{t,j}, w)^{p_1} \right)^{\frac{1}{p_1}},$$

for all $z_{t,j} \in Z_t$, all $t = 1, \dots, k$, all $j_t = 1, \dots, m_t$ and $m_t \in \mathbb{N}$. Then

$$\left(\sup_{v \in V} \sum_{j_1=1}^{m_1} \cdots \sum_{j_k=1}^{m_k} S(z_{1,j_1}, \dots, z_{k,j_k}, v)^{\frac{p_1 p_2}{kp_1 - (k-1)p_2}} \right)^{\frac{kp_1 - (k-1)p_2}{p_1 p_2}} \leq C \prod_{t=1}^k \left(\sup_{w \in W_t} \sum_{j=1}^{m_t} R_t(z_{t,j}, w)^{p_2} \right)^{\frac{1}{p_2}},$$

for all $z_{t,j} \in Z_t$, all $t = 1, \dots, k$, all $j_t = 1, \dots, m_t$ and $m_t \in \mathbb{N}$.

We omit the details of the proof of Theorem 2.2. A careful scrutiny of the second part of the proof of Theorem 2.1 and Theorem 2.2 yields a useful regularity principle itself for anisotropic summability of sequences. As we shall apply such an estimate in the upcoming sections, we state it as a separate Theorem.

Theorem 2.3 (Anisotropic Regularity Principle). *Let $p_1, p_2, r_1, r_2 \geq 1$ and $p_3 \geq p_1$ and $r_3 \geq r_1$ with*

$$\frac{1}{r_1} - \frac{1}{p_1} \leq \frac{1}{r_3} - \frac{1}{p_3}.$$

Then

$$\begin{aligned} & \sup_{v \in V} \left(\sum_{i=1}^{m_1} \left(\sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{p_2} \right)^{\frac{1}{p_2} p_1} \right)^{\frac{1}{p_1}} \\ & \leq C \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{r_1} \right)^{\frac{1}{r_1}} \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{r_2} \right)^{\frac{1}{r_2}}, \end{aligned}$$

for all $z_{1,i}, z_{2,j}$ and all $m_1, m_2 \in \mathbb{N}$ implies

$$\begin{aligned} & \left(\sup_{v \in V} \left(\sum_{i=1}^{m_1} \left(\sum_{j=1}^{m_2} S(z_{1,i}, z_{2,j}, v)^{p_2} \right)^{\frac{1}{p_2} p_3} \right)^{\frac{1}{p_3}} \right) \\ & \leq C \left(\sup_{w \in W_1} \sum_{i=1}^{m_1} R_1(z_{1,i}, w)^{r_3} \right)^{\frac{1}{r_3}} \left(\sup_{w \in W_2} \sum_{j=1}^{m_2} R_2(z_{2,j}, w)^{r_2} \right)^{\frac{1}{r_2}} \end{aligned}$$

for all $z_{1,i}, z_{2,j}$ and $m_1, m_2 \in \mathbb{N}$.

For the applications we shall carry on in the next sections, S will be constant in V and W_1, \dots, W_k will be compact sets.

3 New inclusion theorems for multiple summing operators

It is well known that every absolutely p -summing linear operator is absolutely q -summing whenever $1 \leq p \leq q$ (see [10]). More generally, absolutely $(p_1; p_2)$ -summing operators are absolutely $(q_1; q_2)$ -summing whenever $1 \leq p_j \leq q_j$, $j = 1, 2$, and

$$\frac{1}{p_2} - \frac{1}{p_1} \leq \frac{1}{q_2} - \frac{1}{q_1}.$$

These kind of results are called inclusion results. For multilinear operators inclusion results are more challenging. For instance, every multiple p -summing multilinear operator is multiple q -summing whenever $1 \leq p \leq q \leq 2$, but this is not valid beyond the threshold 2 (see [22, 23]). In this section, as a consequence of the regularity principle, we provide new inclusion theorems for multiple summing operators.

Henceforth E, E_1, \dots, E_m, F denote Banach spaces over \mathbb{K} . Following classical terminology, the Banach space of all bounded m -linear operators from $E_1 \times \dots \times E_m$ to F is denoted by $\mathcal{L}(E_1, \dots, E_m; F)$ and we endow it with the classical sup norm. The topological dual of E is denoted by E^* and its closed unit ball is denoted by B_{E^*} . Throughout the paper, for $p \in [1, \infty]$, p^* denotes the conjugate of p , that is

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

The convention $1^* = \infty$ and $\infty^* = 1$ will be adopted. Also, as usual, we consider the Banach spaces of weakly and strongly p -summable sequences:

$$\ell_p^w(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \|(x_j)_{j=1}^\infty\|_{w,p} := \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^\infty |\varphi(x_j)|^p \right)^{1/p} < \infty \right\}$$

and

$$\ell_p(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \|(x_j)_{j=1}^\infty\|_p := \left(\sum_{j=1}^\infty \|x_j\|^p \right)^{1/p} < \infty \right\}.$$

For $\mathbf{q} := (q_1, \dots, q_m) \in [1, \infty)^m$, we define the space of m -matrices $\ell_{\mathbf{q}}(E)$ as

$$\ell_{\mathbf{q}}(E) := \ell_{q_1}(\ell_{q_2}(\dots(\ell_{q_m}(E))\dots)).$$

That is, a vector matrix $(x_{i_1 \dots i_m})_{i_1, \dots, i_m=1}^\infty$ belongs to $\ell_{\mathbf{q}}(E)$ if, and only if,

$$\|(x_{i_1 \dots i_m})_{i_1, \dots, i_m=1}^\infty\|_{\ell_{\mathbf{q}}(E)} := \left(\sum_{i_1=1}^\infty \left(\dots \left(\sum_{i_m=1}^\infty \|x_{i_1 \dots i_m}\|_E^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} < \infty.$$

When $E = \mathbb{K}$, we simply write $\ell_{\mathbf{q}}$. The following definition will be useful for our purposes.

Definition 3.1. Let $\mathbf{p} = (p_1, \dots, p_m)$, $\mathbf{q} = (q_1, \dots, q_m) \in [1, \infty]^m$. A multilinear operator $T: E_1 \times \dots \times E_m \rightarrow F$ is said to be multiple $(q_1, \dots, q_m; p_1, \dots, p_m)$ -summing if there exists a constant $C > 0$ such that

$$\left(\sum_{j_1=1}^{\infty} \left(\dots \left(\sum_{j_m=1}^{\infty} \|T(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)})\|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \prod_{k=1}^m \left\| (x_{j_k}^{(k)})_{j_k=1}^{\infty} \right\|_{w, p_k} \quad (6)$$

for all $(x_{j_k}^{(k)})_{j_k=1}^{\infty} \in \ell_{p_k}^w(E_k)$. We represent the class of all multiple $(q_1, \dots, q_m; p_1, \dots, p_m)$ -summing operators by $\Pi_{(q_1, \dots, q_m; p_1, \dots, p_m)}^m(E_1, \dots, E_m; F)$. When $q_j = \infty$, the respective sum is replaced by the sup norm.

The infimum of all $C > 0$ for which (6) holds defines a complete norm, denoted hereafter by $\pi_{(q_1, \dots, q_m; p_1, \dots, p_m)}(\cdot)$. It is not hard to verify that

$$\Pi_{(q_1, \dots, q_m; p_1, \dots, p_m)}^m(E_1, \dots, E_m; F)$$

is a subspace of $\mathcal{L}(E_1, \dots, E_m; F)$ and $\|\cdot\| \leq \pi_{(q_1, \dots, q_m; p_1, \dots, p_m)}(\cdot)$. Also, if $q_j < p_j$ for some $j \in \{1, \dots, m\}$, then

$$\Pi_{(q_1, \dots, q_m; p_1, \dots, p_m)}^m(E_1, \dots, E_m; F) = \{0\}.$$

The following result associates multiple summing operators and Hardy–Littlewood inequality.

Theorem 3.2. Let $(p_1, \dots, p_m) \in [1, \infty]^m$. The following statements are equivalent:

- (1) There is a constant $C > 0$ such that for every $T \in \mathcal{L}(\ell_{p_1}, \dots, \ell_{p_m}; F)$ the following holds

$$\left(\sum_{j_1=1}^{\infty} \left(\dots \left(\sum_{j_m=1}^{\infty} \|T(e_{j_1}, \dots, e_{j_m})\|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \|T\|.$$

- (2) For all Banach spaces E_1, \dots, E_m , we have

$$\mathcal{L}(E_1, \dots, E_m; F) = \Pi_{(q_1, \dots, q_m; p_1^*, \dots, p_m^*)}^{mult}(E_1, \dots, E_m; F).$$

Theorem 3.2, as stated here, is essentially due to D. Pérez-García and I. Villanueva, see [24, Corollary 20], and its proof rests on the isometric isomorphisms $\mathcal{L}(\ell_{p^*}, E) \sim \ell_p^w(E)$ and $\mathcal{L}(c_0, E) \sim \ell_1^w(E)$. An advantage of this result for our purposes in subsequent sections is that it provides a useful way to link Hardy-Littlewood type inequalities to the language of multiple summing operators; for results on multilinear summing operators we refer to [7, 17] and references therein.

The first application of Theorem 2.2 is an inclusion result for multiple summing operators which complements, to some extent, the one from [22]:

Proposition 3.3. *Let $m \geq 2$ be a positive integer and*

$$2 \leq r \leq u < \frac{mr}{m-1}.$$

Then, for any collection of Banach spaces E_1, \dots, E_m, F there holds

$$\Pi_{(r,r)}^m(E_1, \dots, E_m; F) \subset \Pi_{(\frac{ru}{mr-(m-1)u}; u)}^m(E_1, \dots, E_m; F)$$

and the inclusion has norm 1.

Proof. Using the abstract environment of Section 2, we just need to define $Z_j = E_j$; let also $V = \{0\}$ and $T \in \Pi_{(r,r)}^m(E_1, \dots, E_m; F)$. Now define

$$\begin{aligned} W_j &= B_{E_j^*}, \\ R_j(x, \varphi) &= |\varphi(x)|, \\ S(x_1, \dots, x_m, v) &= |T(x_1, \dots, x_m)| \end{aligned}$$

and the proof is a consequence of the Regularity Principle for k -variables (Theorem 2.2). \square

If one carries out the same reasoning employed in the second part of the proof of the Regularity Principle (Theorem 2.1), the following more general result can be established:

Proposition 3.4 ((Inclusion Theorem)). *Let m be a positive integer and $1 \leq s \leq u < \frac{mrs}{mr-s}$. Then, for any Banach spaces E_1, \dots, E_m, F we have*

$$\Pi_{(r,s)}^m(E_1, \dots, E_m; F) \subset \Pi_{(\frac{rsu}{su+mrs-mru}; u)}^m(E_1, \dots, E_m; F)$$

and the inclusion has norm 1.

Proposition 3.4 itself has an interesting application. It provides a simplified proof of a key technical tool from [21], that is: for any positive integer m , and any $p > 2m$, there holds

$$\left(\sum_{i_1, \dots, i_m=1}^{\infty} |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2p}{p-2m}} \right)^{\frac{p-2m}{2p}} \leq \|U\|, \quad (7)$$

for all m -linear forms $U: \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n .

Indeed, as every continuous m -linear form T is multiple $(2; 1)$ -summing with constant 1, one simply takes $(r, s, u) = (2, 1, p^*)$ in the statement of Proposition 3.4 and arrives at (7).

As a matter of fact, every continuous m -linear form T is actually multiple

$$(2, \dots, 2; 1, \dots, 1, 2)\text{-summing}$$

with constant 1. If one uses this stronger information, one can actually improves (7) as it yields the $\ell_{\frac{2p}{p-2m+2}}$ -norm of $|U(e_{i_1}, \dots, e_{i_m})|$ is controlled by $\|U\|$.

4 Grothendieck-type theorems

Every continuous linear operator from ℓ_1 into ℓ_2 is absolutely (q, p) -summing for every $q \geq p \geq 1$; this result is a trademark theorem proven by Grothendieck in his seminal 1950's "Résumé", [13] — for recent monographs on Grothendieck's Résumé we refer to [9, 25]. More precisely, the result asserts that

$$\left(\sum_{j=1}^m \|u(x_j)\|^q \right)^{\frac{1}{q}} \leq C \sup_{\varphi \in B_{\ell_\infty}} \left(\sum_{j=1}^m |\varphi(x_j)|^p \right)^{\frac{1}{p}}$$

for all continuous linear operators $u: \ell_1 \rightarrow \ell_2$. This result is in fact very special as illustrated by the following result due to Lindenstrauss and Pełczyński ([15]): if E, F are infinite dimensional Banach spaces and E has unconditional Schauder basis, and every continuous linear operator from E to F is absolutely 1-summing, then $E = \ell_1$ and F is a Hilbert space. In the multilinear setting, every continuous m -linear operator from $\ell_1 \times \cdots \times \ell_1$ into ℓ_2 is multiple (q, p) -summing for every $1 \leq p \leq 2$ and every $q \geq p$ ([7, Theorems 5.1 and 5.2] and [22]), and, when $m \geq 2$, this result is no longer valid if $p > 2$ ([23, Theorem 3.6]). The results of the previous section provide estimates for values of q for which every continuous m -linear operator $T: \ell_1 \times \cdots \times \ell_1 \rightarrow \ell_2$ is multiple (q, p) -summing when $p = 2 + \epsilon$, for ϵ small. However, since (ℓ_1, ℓ_2) is a quite special pair of Banach spaces for summability purposes, we are able to provide a definitive result with all pairs of (q, p) for which $\Pi_{(q,p)}^m({}^m\ell_1; \ell_2) = \mathcal{L}({}^m\ell_1; \ell_2)$.

Theorem 4.1. *Let $m \geq 2$ be a positive integer and $1 \leq p \leq q < \infty$. Then*

$$\Pi_{(q,p)}^m({}^m\ell_1; \ell_2) = \mathcal{L}({}^m\ell_1; \ell_2)$$

if and only if $p \leq 2$ or $q > p > 2$.

Proof. If $p \leq 2$, by [22] we know that every continuous m -linear operator is multiple (q, p) -summing for all $q \geq p$. If $p > 2$, by [23] we know that $\Pi_{(p,p)}^m({}^m\ell_1; \ell_2) \neq \mathcal{L}({}^m\ell_1; \ell_2)$. It remains to prove that $\Pi_{(q,p)}^m({}^m\ell_1; \ell_2) = \mathcal{L}({}^m\ell_1; \ell_2)$ for all $q > p > 2$. So, let us consider $q > p > 2$. By [8, Proposition 3.6] we know that

$$\Pi_{(q,p)}^m({}^m\ell_1; \mathbb{K}) = \mathcal{L}({}^m\ell_1; \mathbb{K}). \quad (8)$$

It is not difficult to prove that from (8) we conclude that every continuous m -linear operator $T: \ell_1 \times \cdots \times \ell_1 \rightarrow F$ sends weakly p -summable sequences into weakly q -summable sequences, regardless of the Banach space F . More precisely,

$$\left(T(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}) \right)_{j_1, \dots, j_m=1}^\infty \in \ell_q^w(F)$$

whenever

$$(x_{j_k}^{(k)})_{j_k=1}^\infty \in \ell_p^w(E_k), \quad k = 1, \dots, m.$$

Now, considering $\Psi: \ell_1 \times \cdots \times \ell_1 \rightarrow \ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1$ given by

$$\Psi(x^{(1)}, \dots, x^{(m)}) = x^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x^{(m)},$$

we conclude that

$$\left(x_{j_1}^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_{j_m}^{(m)}\right)_{j_1, \dots, j_m=1}^\infty \in \ell_q^w(\ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1)$$

whenever

$$\left(x_{j_k}^{(k)}\right)_{j_k=1}^\infty \in \ell_p^w(\ell_1), \quad k = 1, \dots, m.$$

Let $\widetilde{T} : \ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1 \rightarrow \ell_2$ be the linearization of T . Since $\ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1$ is isometrically isomorphic to ℓ_1 , then \widetilde{T} is absolutely q -summing and, for $\left(x_{j_k}^{(k)}\right)_{j_k=1}^\infty \in \ell_p^w(\ell_1)$, $k = 1, \dots, m$, we have

$$\begin{aligned} \left(\sum_{j_1, \dots, j_m=1}^\infty \left\|T\left(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}\right)\right\|^q\right)^{\frac{1}{q}} &= \left(\sum_{j_1, \dots, j_m=1}^\infty \left\|\widetilde{T}\left(x_{j_1}^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_{j_m}^{(m)}\right)\right\|^q\right)^{\frac{1}{q}} \\ &\leq C \left\|\left(x_{j_1}^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_{j_m}^{(m)}\right)_{j_1, \dots, j_m=1}^\infty\right\|_{w, q} < \infty \end{aligned}$$

and the proof is done. \square

As a matter of fact a similar result holds in a more general setting:

Theorem 4.2. *Let $m \geq 2$ be an integer and F be a Banach space. If $\Pi_{(q; p)}(\ell_1; F) = \mathcal{L}(\ell_1; F)$, then*

$$\Pi_{(q+\delta; p)}^m({}^m\ell_1; F) = \mathcal{L}({}^m\ell_1; F)$$

for all $\delta > 0$.

Proof. By [8, Proposition 3.6] we know that

$$\Pi_{(p+\varepsilon; p)}^m({}^m\ell_1; \mathbb{K}) = \mathcal{L}({}^m\ell_1; \mathbb{K})$$

for all $\varepsilon > 0$. As in the previous proof we know that

$$\left(x_{j_1}^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_{j_m}^{(m)}\right)_{j_1, \dots, j_m=1}^\infty \in \ell_{p+\varepsilon}^w(\ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1)$$

whenever

$$\left(x_{j_k}^{(k)}\right)_{j_k=1}^\infty \in \ell_p^w(\ell_1), \quad k = 1, \dots, m.$$

Let $\widetilde{T} : \ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1 \rightarrow F$ be the linearization of T . Since $\ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1$ is isometrically isomorphic to ℓ_1 , then \widetilde{T} is absolutely $(q; p)$ -summing and hence for any $\delta > 0$ there is a $\varepsilon > 0$ such that \widetilde{T} is $(q + \delta; p + \varepsilon)$ -summing. Therefore, for $\left(x_{j_k}^{(k)}\right)_{j_k=1}^\infty \in \ell_p^w(\ell_1)$, $k = 1, \dots, m$, we have

$$\begin{aligned} \left(\sum_{j_1, \dots, j_m=1}^\infty \left\|T\left(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}\right)\right\|^{q+\delta}\right)^{\frac{1}{q+\delta}} &= \left(\sum_{j_1, \dots, j_m=1}^\infty \left\|\widetilde{T}\left(x_{j_1}^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_{j_m}^{(m)}\right)\right\|^{q+\delta}\right)^{\frac{1}{q+\delta}} \\ &\leq C \left\|\left(x_{j_1}^{(1)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_{j_m}^{(m)}\right)_{j_1, \dots, j_m=1}^\infty\right\|_{w, p+\varepsilon} < \infty \end{aligned}$$

and the proof is complete. \square

5 Sharp anisotropic Hardy–Littlewood inequality

The investigation of bilinear forms acting on sequence spaces goes back to the pioneering work of Hilbert on his famous double-series theorem and, throughout the 20th century, has attracted the attention of leading mathematicians as Weyl, Toeplitz, Schur, Nehari, see [18], [28], [29] and references therein. A trademark of the field comes from Littlewood’s solution, [16], to the problem posed by P.J. Daniell; the famous Littlewood’s $4/3$ inequality is an estimate that represents the extremal case $p = q = \infty$ in (3). Bohnenblust and Hille, [6], obtained important generalizations of Littlewood’s $4/3$ inequality to the setting of m -linear operators and few years later, Hardy and Littlewood proved a series of inequalities for bilinear forms acting on $\ell_p \times \ell_q$ spaces, with $\frac{1}{p} + \frac{1}{q} < 1$, which would launch a new and promising line of investigation; named thereafter Hardy-Littlewood type inequalities. The key objective of study is to control

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \quad (9)$$

for all norm-1 bilinear operators $T : \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$ and all positive integers n .

The search of optimal ranges of exponents for universal summability of (9) has been carried out by direct and indirect approaches permeating the theory and some sectional answers have been collected throughout the decades. Partial solutions can be found, for instance, in [2] and [19]. While these represented important advances in the global understanding of the problem, the results proven thus far are limited when it comes to determining the whole spectrum of admissible exponents. This is, indeed, a subtle issue which resembles the problem of Schur multipliers investigated by Bennett in [4].

Before we state the main Theorem of this section, we highlight that this is more than just a beautiful mathematical puzzle. Even when restricted to *classical* isotropic multiple summing, enlarging the studies to the anisotropic setting reveals a number of important nuances that could not be perceived otherwise. This is a somewhat common procedure in the realm of mathematics — solving real problems through complex methods being probably the most emblematic example.

As an application of the regularity principle, we will prove the following complete characterization of all admissible anisotropic exponents for the Hardy-Littlewood inequality:

Theorem 5.1. *Let $p, q \in [2, \infty]$ with $\frac{1}{p} + \frac{1}{q} < 1$, and $a, b > 0$. The following assertions are equivalent:*

(a) *There is a constant $C_{p,q,a,b} \geq 1$ such that*

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |U(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \leq C_{p,q,a,b} \|U\|, \quad (10)$$

for all bilinear operators $U: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$ and all positive integers n .

(b) The exponents a, b satisfy $(a, b) \in [\frac{q}{q-1}, \infty) \times [\frac{pq}{pq-p-q}, \infty)$ and

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right). \quad (11)$$

The proof of Theorem 5.1 will be developed in the sequel. As commented above, the main technical novelty of the proof is the regularity principle established in Section 2, which, in this particular case, reveals sharp and subtle inclusion properties that were not accessible by preceding methods.

We start off by recalling the following inequality sometimes credited to Minkowski (see, for instance, [12]): if $1 \leq p \leq q$, then

$$\left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^p \right)^{\frac{1}{p} \cdot q} \right)^{\frac{1}{q}} \leq \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}|^q \right)^{\frac{1}{q} \cdot p} \right)^{\frac{1}{p}} \quad (12)$$

for all sequence of scalars matrices (a_{ij}) . We will also make use of the main result from [2], which we state here for the readers' convenience. From now on

$$\mathbf{p} := (p_1, \dots, p_m) \in [1, \infty]^m$$

and we denote

$$\left| \frac{1}{\mathbf{p}} \right| := \frac{1}{p_1} + \dots + \frac{1}{p_m}.$$

Theorem 5.2 (Generalized Hardy–Littlewood inequality [2]). *Let $\mathbf{p} := (p_1, \dots, p_m) \in [1, \infty]^m$ be such that*

$$\left| \frac{1}{\mathbf{p}} \right| \leq \frac{1}{2} \quad \text{and} \quad \mathbf{q} := (q_1, \dots, q_m) \in \left[\left(1 - \left| \frac{1}{\mathbf{p}} \right| \right)^{-1}, 2 \right]^m.$$

The following are equivalent:

(1) *There is a constant $C_{m, \mathbf{p}, \mathbf{q}}^{\mathbb{K}} \geq 1$ such that*

$$\left(\sum_{j_1=1}^{\infty} \left(\dots \left(\sum_{j_m=1}^{\infty} \left| A(e_{j_1}, \dots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{m, \mathbf{p}, \mathbf{q}}^{\mathbb{K}} \|A\| \quad (13)$$

for all m -linear forms $A: \ell_{p_1}^n \times \dots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ and all positive integers n .

(2) *The inequality*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left| \frac{1}{\mathbf{p}} \right|$$

is verified.

We begin with a Lemma which plays a key role in the solution of the classification problem for all sharp exponents in the anisotropic Hardy–Littlewood inequality. In the heart of its proof lies the Regularity Principle established in Section 2.

Lemma 5.3. *Let E_1, E_2 be Banach spaces, $p \in (2, \infty)$, and $q \in [2, \infty]$. Then every continuous 2-linear operator $U : E_1 \times E_2 \rightarrow \mathbb{K}$ is multiple $\left(\frac{2p}{p-2}, \frac{q}{q-1}; p^*, q^*\right)$ -summing.*

Proof. Initially we observe that from Theorem 5.2 and Theorem 3.2 there is a constant C_0 such that

$$\left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |U(x_i, y_j)|^2 \right)^{\frac{1}{2} \cdot 1} \right)^{\frac{1}{1}} \leq C_0 \|U\| \| (x_i) \|_{w,1} \| (y_j) \|_{w,1}.$$

By the Anisotropic Regularity Principle we have

$$\left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |U(x_i, y_j)|^2 \right)^{\frac{1}{2} \cdot q^*} \right)^{\frac{1}{q^*}} \leq C_0 \|U\| \| (x_i) \|_{w,1} \| (y_j) \|_{w,q^*}.$$

Since $q^* \leq 2$, by the Minkowski inequality (12) there holds

$$\left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |U(x_i, y_j)|^{q^*} \right)^{\frac{1}{q^*} \cdot 2} \right)^{\frac{1}{2}} \leq C_0 \|U\| \| (x_i) \|_{w,1} \| (y_j) \|_{w,q^*}.$$

Finally, by the Anisotropic Regularity Principle we obtain

$$\left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |U(x_i, y_j)|^{q^*} \right)^{\frac{1}{q^*} \cdot \frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \leq C_0 \|U\| \| (x_i) \|_{w,p^*} \| (y_j) \|_{w,q^*},$$

and the Lemma is proven. \square

Next, we will make use of a Hölder-type inequality essentially due to Benedek and Panzone [3] and a generalization of the Kahane–Salem–Zygmund inequality; to assist the readers, we state them both as we will need and cite [1, page 50] and [2], [5] for their proofs.

Theorem 5.4 (Interpolative Hölder inequality). *Let n be a positive integer and*

$$q_1, q_2, q_1(k), q_2(k) \in [1, \infty]$$

with $k = 1, 2$ be such that

$$\left(\frac{1}{q_1}, \frac{1}{q_2} \right) = \theta \left(\frac{1}{q_1(1)}, \frac{1}{q_2(1)} \right) + (1 - \theta) \left(\frac{1}{q_1(2)}, \frac{1}{q_2(2)} \right)$$

for a certain $\theta \in [0, 1]$. Then

$$\begin{aligned} & \left(\sum_{i_1=1}^n \left(\sum_{i_2=1}^n |a_{i_1, i_2}|^{q_2} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \\ & \leq \left[\left(\sum_{i_1=1}^n \left(\sum_{i_2=1}^n |a_{i_1, i_2}|^{q_2(1)} \right)^{\frac{q_1(1)}{q_2(1)}} \right)^{\frac{1}{q_1(1)}} \right]^{\theta} \cdot \left[\left(\sum_{i_1=1}^n \left(\sum_{i_2=1}^n |a_{i_1, i_2}|^{q_2(2)} \right)^{\frac{q_1(2)}{q_2(2)}} \right)^{\frac{1}{q_1(2)}} \right]^{1-\theta}, \end{aligned}$$

for all positive integers n .

Theorem 5.5 (Kahane–Salem–Zygmund inequality). *Let $m, n \geq 1$ and $p_1, \dots, p_m \in [2, \infty]$. There is an universal constant C_m , depending only on m , and an m -linear mapping $A_n: \ell_{p_1}^n \times \dots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ of the form*

$$A_n(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^n \pm z_{i_1}^{(1)} \dots z_{i_m}^{(m)}$$

such that

$$\|A_n\| \leq C_m n^{\frac{m+1}{2} - \left(\frac{1}{p_1} + \dots + \frac{1}{p_m}\right)}.$$

We have gathered all the tools needed to deliver a proof of the main result of this section classifying all possible exponents $a, b > 0$ for which there is a Hardy–Littlewood-type inequality for bilinear forms $U: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$ with $p, q \in [2, \infty]$ and $\frac{1}{p} + \frac{1}{q} < 1$:

Proof of Theorem 5.1.

We will divide our analysis in two cases: when $p > 2$ and when $p = 2$. Let us start the proof in the case $p > 2$.

(b) \Rightarrow (a). Suppose that $(a, b) \in [\frac{q}{q-1}, 2] \times [\frac{pq}{pq-p-q}, \frac{2p}{p-2}]$ with

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right). \quad (14)$$

It suffices to consider the case in which we have an equality in (14). There is a $\theta \in [0, 1]$ such that

$$\left(\frac{1}{b}, \frac{1}{a} \right) = \theta \left(\frac{p-2}{2p}, \frac{q-1}{q} \right) + (1-\theta) \left(\frac{1}{\lambda}, \frac{1}{2} \right),$$

where $\lambda = \frac{pq}{pq-p-q}$. Applying Theorem 5.4, Lemma 5.3, combined with [14, Theorems

1 and 2], we conclude that there are constants $C_0, C_1 \geq 1$ such that

$$\begin{aligned}
& \left(\sum_{i=1}^n \left(\sum_{j=1}^n |U(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \\
& \leq \left[\left(\sum_{i=1}^n \left(\sum_{j=1}^n |U(e_i, e_j)|^{q^*} \right)^{\frac{1}{q^*} \cdot \frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \right]^{\theta} \cdot \left[\left(\sum_{i=1}^n \left(\sum_{j=1}^n |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]^{1-\theta} \\
& \leq (C_0 \|U\|)^{\theta} \cdot (C_1 \|U\|)^{1-\theta} \\
& = (C_0)^{\theta} C_1^{1-\theta} \|U\|.
\end{aligned}$$

The case $(a, b) \in \left(\left[\frac{q}{q-1}, 2 \right] \times \left[\frac{2p}{p-2}, \infty \right) \right) \cup \left([2, \infty) \times \left[\frac{pq}{pq-p-q}, \infty \right) \right)$ is a straightforward consequence of the previous result for $\theta = 0$ and $\theta = 1$.

(a) \Rightarrow (b). For any positive integer n , consider the bilinear operator

$$U_n: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$$

given by

$$U_n(x, y) = x_1 \sum_{j=1}^n y_j.$$

Since $\|U_n\| = n^{\frac{1}{q}}$, plugging U_n into (10) we conclude that

$$n^{\frac{1}{a}} \leq C_{p,q} n^{\frac{1}{q}}$$

for a certain constant $C_{p,q}$, and since n is arbitrary we conclude that $a \geq \frac{q}{q-1}$. Now we consider the bilinear operator

$$V_n: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$$

given by

$$V_n(x, y) = \sum_{j=1}^n x_j y_j.$$

Since $\|V_n\| = n^{1-(\frac{1}{p}+\frac{1}{q})}$, plugging V_n into (10) we conclude that

$$n^{\frac{1}{b}} \leq C_{p,q} n^{1-(\frac{1}{p}+\frac{1}{q})}$$

for a certain constant $C_{p,q}$, and thus

$$b \geq \frac{pq}{pq-p-q}.$$

It remains to verify that for $(a, b) \in \left[\frac{q}{q-1}, \infty \right) \times \left[\frac{pq}{pq-p-q}, \infty \right)$ the exponents must obey (11). Let $A_n: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$ be the bilinear form given by the Kahane–Salem–Zygmund inequality. Using (10) with A_n we obtain

$$n^{\frac{1}{a}+\frac{1}{b}} \leq n^{\frac{3}{2}-(\frac{1}{p}+\frac{1}{q})},$$

and thus

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right)$$

This concludes the proof of Theorem 5.1 when $p > 2$.

Let us turn our attention to the case $p = 2$. Initially, we revisit the proof of Lemma 5.3 and note that if E_1, E_2 are Banach spaces and $q \in [2, \infty)$, then every continuous 2-linear operator $T: E_1 \times E_2 \rightarrow \mathbb{K}$ is multiple $(\infty, q^*; 2, q^*)$ -summing. Thus, there is a constant $C_0 \geq 1$ such that

$$\left(\sup_i \left(\sum_{j=1}^n |U(e_i, e_j)|^{q^*} \right)^{\frac{1}{q^*}} \right) \leq C_0 \|U\|, \quad (15)$$

for all $U: \ell_2^n \times \ell_q^n \rightarrow \mathbb{K}$ and all positive integers n . We proceed with the proof.

(b) \Rightarrow (a). Suppose that $(a, b) \in (\frac{q}{q-1}, 2] \times [\frac{2q}{q-2}, \infty)$ with

$$\frac{1}{a} + \frac{1}{b} \leq \frac{1}{q^*}. \quad (16)$$

It suffices to consider the case in which we have an equality in (16). We can find $\theta \in [0, 1)$ such that

$$\left(\frac{1}{b}, \frac{1}{a} \right) = \theta \left(\frac{1}{\infty}, \frac{q-1}{q} \right) + (1-\theta) \left(\frac{q-2}{2q}, \frac{1}{2} \right).$$

By Theorem 5.4, [14, Theorem 2] and (15) there exist constants C_0, C_1 such that

$$\begin{aligned} & \left(\sum_{i=1}^n \left(\sum_{j=1}^n |U(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \\ & \leq \left[\left(\sup_i \left(\sum_{j=1}^n |U(e_i, e_j)|^{q^*} \right)^{\frac{1}{q^*}} \right)^\theta \right]^\theta \cdot \left[\left(\sum_{i=1}^n \left(\sum_{j=1}^n |U(e_i, e_j)|^2 \right)^{\frac{1}{2} \cdot \frac{2q}{q-2}} \right)^{\frac{q-2}{2q}} \right]^{1-\theta} \\ & \leq (C_0 \|U\|)^\theta \cdot (C_1 \|U\|)^{1-\theta} \\ & = (C_0)^\theta C_1^{1-\theta} \|U\|. \end{aligned}$$

The case $(a, b) \in [2, \infty) \times [\frac{2q}{q-2}, \infty)$ is a straightforward consequence of the previous result for $\theta = 0$. The proof of (a) \Rightarrow (b) is a consequence of the Kahane–Salem–Zygmund inequality as before. This concludes the proof of Theorem 5.1. \square

We conclude this section commenting on the case where the sums are in the reverse order. Arguing by symmetry, the following also holds: let $p \in [2, \infty]$, $q \in [2, \infty]$ with $\frac{1}{p} + \frac{1}{q} < 1$, and $a, b > 0$. The following assertions are equivalent:

(a) There is a constant $C \geq 1$ such that

$$\left(\sum_{j=1}^n \left(\sum_{i=1}^n |U(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \leq C \|U\|$$

for all bilinear operators $U : \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$ and all positive integers n .

(b) The exponents a, b satisfy $(a, b) \in [\frac{p}{p-1}, \infty) \times [\frac{pq}{pq-p-q}, \infty)$ with

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right).$$

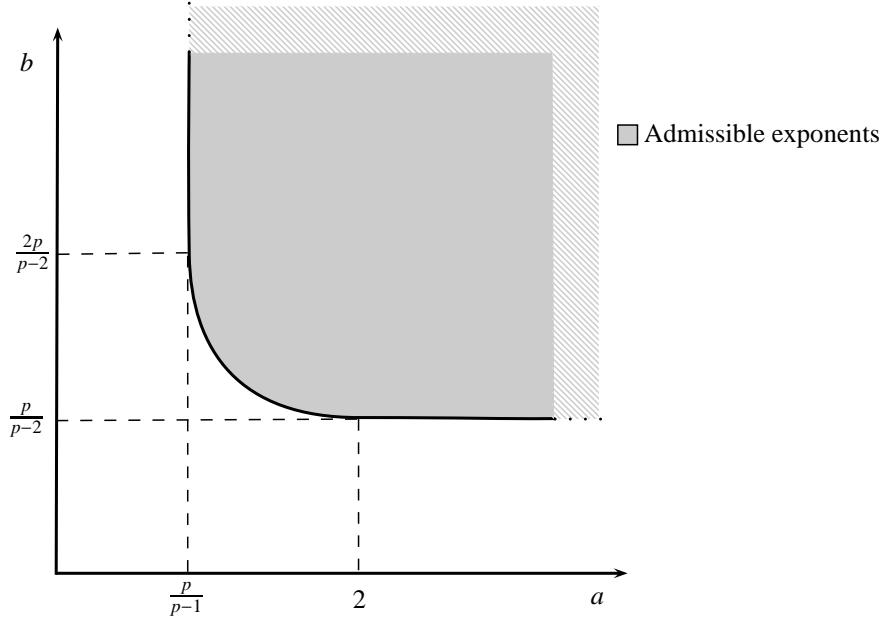


Figure 1: Plot of the region of all admissible anisotropic exponents (a, b) for which the Hardy-Littlewood inequality remains universally bounded in the case $p = q$. The curve joining the points $(\frac{p}{p-1}, \frac{2p}{p-2})$ and $(2, \frac{p}{p-2})$ is the hyperbola $b = \frac{2p \cdot a}{(3p-4)a-2p}$.

6 Dimension blow-up

Theorem 5.1 is a complete classification of all possible anisotropic Hardy-Littlewood type estimates. In this Section we turn our attention to the order in which the estimates blow-up with respect to the dimension in the cases when the exponents (a, b) are out

of the range predicted by Theorem 5.1. In this scenario, there is no such a constant C , independent of dimension n , for which the inequality

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^a \right)^{\frac{1}{a} \cdot b} \right)^{\frac{1}{b}} \leq C \cdot \|T\| \quad (17)$$

is satisfied universally for all bilinear operators $T: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K}$. Our goal is to obtain the precise dependence arising on n .

Hereafter in this Section, we denote a non Hardy–Littlewood pair of exponents by (r_1, r_2) and divide the region occupied by the non Hardy–Littlewood exponents (r_1, r_2) in four sub-regions:

(R1) (r_1, r_2) such that $q^* \leq r_1 \leq 2$ and

$$\frac{1}{r_1} + \frac{1}{r_2} > \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right).$$

(R2) (r_1, r_2) such that $r_1 < q^*$ and $r_2 < \frac{2p}{p-2}$.

(R3) (r_1, r_2) such that $r_1 < q^*$ and $r_2 > \frac{2p}{p-2}$.

(R4) (r_1, r_2) such that $r_1 > 2$ and $r_2 < \frac{pq}{pq-p-q}$.

We will also use the following notation:

$$\mathcal{B}_{p,q}^n := \left\{ T: \ell_p^n \times \ell_q^n \rightarrow \mathbb{K} : T \text{ is a bilinear form and } \|T\| = 1 \right\}.$$

Proposition 6.1. *Let $p \in (2, \infty]$, $q \in [2, \infty]$.*

(i) *If (r_1, r_2) are in (R1) or in (R2), then*

$$\sup_{\mathcal{B}_{p,q}^n} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} = O \left(n^{\frac{1}{r_1} + \frac{1}{r_2} - \left(\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right) \right)} \right),$$

and the exponent $\frac{1}{r_1} + \frac{1}{r_2} - \left(\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right) \right)$ is optimal.

(ii) *If (r_1, r_2) are in (R3), then*

$$\sup_{\mathcal{B}_{p,q}^n} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} = O \left(n^{\frac{1}{r_1} - \frac{1}{q^*}} \right),$$

and the exponent $\frac{1}{r_1} - \frac{1}{q^}$ is optimal.*

(iii) If (r_1, r_2) are in (R4), then

$$\sup_{\mathcal{B}_{p,q}^n} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} = O \left(n^{\frac{1}{r_2} - \frac{pq-p-q}{pq}} \right),$$

and the exponent $\frac{1}{r_2} - \frac{pq-p-q}{pq}$ is optimal.

Proof. (i) Suppose that (r_1, r_2) are in (R1) and let $\delta > 0$ be such that

$$\frac{1}{r_1} + \frac{1}{\delta} = \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right).$$

By Hölder inequality for mixed sums and Theorem 5.1, if

$$\frac{1}{X} + \frac{1}{\delta} = \frac{1}{r_2},$$

then

$$\begin{aligned} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} &\leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \delta} \right)^{\frac{1}{\delta}} \cdot \left(\sum_{i=1}^n \left(\sup_j 1 \right)^X \right)^{\frac{1}{X}} \\ &\leq C \|T\| n^{\frac{1}{X}} \\ &= C \|T\| n^{\frac{1}{r_1} + \frac{1}{r_2} - \left(\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right) \right)}. \end{aligned}$$

To prove the optimality, consider

$$A_n: \ell_p \times \ell_q \rightarrow \mathbb{K}$$

given by the Kahane–Salem–Zygmund inequality. If

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |A_n(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} \leq C n^t \|A_n\|$$

for a certain t , then

$$n^{\frac{1}{r_1} + \frac{1}{r_2}} \leq C n^t n^{\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right)}.$$

Since n is arbitrary, we conclude that

$$t \geq \frac{1}{r_1} + \frac{1}{r_2} - \left(\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right) \right).$$

Let us now consider the case when (r_1, r_2) lies in the region (R2). Let X_1, X_2 be such that

$$\begin{aligned} \frac{1}{r_1} &= \frac{1}{q^*} + \frac{1}{X_1}, \\ \frac{1}{r_2} &= \frac{1}{\frac{2p}{p-2}} + \frac{1}{X_2}. \end{aligned}$$

By the Hölder inequality for mixed sums and Theorem 5.1, we can estimate

$$\begin{aligned}
\left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} &\leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{q^*} \right)^{\frac{1}{q^*} \cdot \frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \cdot \left(\sum_{i=1}^n \left(\sum_{j=1}^n |1|^{X_1} \right)^{\frac{1}{X_1} \cdot X_2} \right)^{\frac{1}{X_2}} \\
&\leq C \|T\| n^{\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{q^*} - \frac{1}{\frac{2p}{p-2}}} \\
&= C \|T\| n^{\frac{1}{r_1} + \frac{1}{r_2} - \left(\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right) \right)}.
\end{aligned}$$

To prove the optimality, consider

$$A_n: \ell_p \times \ell_q \rightarrow \mathbb{K}$$

given by the Kahane–Salem–Zygmund inequality. If

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |A_n(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} \leq C n^t \|A_n\|,$$

for a certain t , then

$$n^{\frac{1}{r_1} + \frac{1}{r_2}} \leq C n^t n^{\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right)}.$$

Since n is arbitrary, we conclude that

$$t \geq \frac{1}{r_1} + \frac{1}{r_2} - \left(\frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right) \right).$$

(ii) By the Hölder inequality for mixed sums and Theorem 5.1, if

$$\frac{1}{q^*} + \frac{1}{\delta} = \frac{1}{r_1},$$

then

$$\begin{aligned}
\left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} &\leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{q^*} \right)^{\frac{1}{q^*} \cdot r_2} \right)^{\frac{1}{r_2}} \cdot \sup_i \left(\sum_{j=1}^n 1^\delta \right)^{\frac{1}{\delta}} \\
&\leq C \|T\| n^{\frac{1}{r_1} - \frac{1}{q^*}}.
\end{aligned}$$

To prove the optimality, consider

$$A_n: \ell_p \times \ell_q \rightarrow \mathbb{K}$$

given by

$$A_n(x, y) = x_1 \sum_{j=1}^n y_j.$$

If

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |A_n(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} \leq C n^t \|A_n\|,$$

for a certain t , then

$$n^{\frac{1}{r_1}} \leq C n^t n^{\frac{1}{q^*}}.$$

Since n is arbitrary, we conclude that

$$t \geq \frac{1}{r_1} - \frac{1}{q^*}.$$

(iii) By the Hölder inequality for mixed sums and Theorem 5.1, if

$$\frac{1}{\frac{pq}{pq-p-q}} + \frac{1}{\delta} = \frac{1}{r_2},$$

then

$$\begin{aligned} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} &\leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot \frac{pq}{pq-p-q}} \right)^{\frac{1}{\frac{pq}{pq-p-q}}} \cdot \left(\sum_{i=1}^n \left(\sup_j 1^\delta \right) \right)^{\frac{1}{\delta}} \\ &\leq C \|T\| n^{\frac{1}{r_2} - \frac{pq-p-q}{pq}}. \end{aligned}$$

To prove the optimality, consider

$$A_n: \ell_p \times \ell_q \rightarrow \mathbb{K}$$

given by

$$A_n(x, y) = \sum_{j=1}^n x_j y_j.$$

If

$$\left(\sum_{i=1}^n \left(\sum_{j=1}^n |A_n(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} \leq C n^t \|A_n\|$$

for a certain t , then

$$n^{\frac{1}{r_2}} \leq C n^t n^{1 - (\frac{1}{p} + \frac{1}{q})}.$$

Since n is arbitrary, we conclude that

$$t \geq \frac{1}{r_2} - \frac{pq - p - q}{pq},$$

and the proposition is proven. \square

The case $p = 2$ involves a simpler analysis.

Proposition 6.2. *Let $q \in (2, \infty]$ and (r_1, r_2) be not a pair of Hardy–Littlewood exponents, then*

$$\sup_{\mathcal{B}_{2,q}^n} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} \leq O \left(n^{\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{q^*}} \right),$$

and the exponent $\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{q^*}$ is optimal.

Proof. If $r_1 < q^*$, we set X to satisfy

$$\frac{1}{X} + \frac{1}{q^*} = \frac{1}{r_1}.$$

From Hölder inequality for mixed sums and (15), we reach

$$\begin{aligned} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} &\leq \sup_i \left(\sum_{j=1}^n |T(e_i, e_j)|^{q^*} \right)^{\frac{1}{q^*}} \cdot \left(\sum_{i=1}^n \left(\sum_{j=1}^n 1^X \right)^{\frac{1}{X} \cdot r_2} \right)^{\frac{1}{r_2}} \\ &\leq C_0 \|T\| n^{\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{q^*}}. \end{aligned}$$

If $r_1 \geq q^*$, we set X and Y through the equalities

$$\begin{aligned} \frac{1}{r_1} + \frac{1}{X} &= \frac{1}{q^*}, \\ \frac{1}{Y} + \frac{1}{X} &= \frac{1}{r_2}. \end{aligned}$$

Applying once more Hölder inequality for mixed sums and Theorem 5.1, we obtain a constant C such that

$$\begin{aligned} \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} &\leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot X} \right)^{\frac{1}{X}} \cdot \left(\sum_{i=1}^n \left(\sup_j 1 \right)^Y \right)^{\frac{1}{Y}} \\ &\leq C \|T\| n^{\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{q^*}}. \end{aligned}$$

The optimality is proved as in Proposition 6.1 □

When $p, q > 1$ are such that

$$\frac{1}{p} + \frac{1}{q} \geq 1,$$

it is easy to verify that there is no admissible exponent for which (17) remains universally bounded. Next, we investigate the dimensional dependence in this case and, for that, it is useful to denote

$$\mathcal{B}_{p,q}^{n_1, n_2} := \left\{ T : \ell_p^{n_1} \times \ell_q^{n_2} \rightarrow \mathbb{K} : T \text{ is a bilinear form and } \|T\| = 1 \right\}.$$

Proposition 6.3. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} \geq 1$. There holds*

$$\sup_{\mathcal{B}_{p,q}^{n_1,n_2}} \left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} |T(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} = O\left(n_1^{\frac{1}{r_2}}\right) \cdot O\left(n_2^{\frac{1}{r_1} - \frac{1}{\max\{r_1, q^*\}}}\right),$$

and such a blow-up rate of n_2 is sharp and the blow up rate of n_1 is sharp when $r_1 \geq q^*$.

Proof. Under the condition $\frac{1}{p} + \frac{1}{q} \geq 1$, it is simple to check that any $T \in \mathcal{B}_{p,q}^{n_1,n_2}$ is multiple $(\infty, \max\{r_1, q^*\}; p^*, q^*)$ -summing. Let us denote $T(e_i, e_j) = T_{ij}$. By Hölder inequality for mixed sums, if

$$\frac{1}{r_1} = \frac{1}{t_1} + \frac{1}{\max\{r_1, q^*\}},$$

then

$$\begin{aligned} \left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} |T_{ij}|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} &\leq \left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} |1|^{t_1} \right)^{\frac{1}{t_1} \cdot r_2} \right)^{\frac{1}{r_2}} \cdot \sup_i \left(\sum_{j=1}^{n_2} |T_{ij}|^{\max\{r_1, q^*\}} \right)^{\frac{1}{\max\{r_1, q^*\}}} \\ &\leq C \|T\| n_1^{\frac{1}{r_2}} n_2^{\frac{1}{r_1} - \frac{1}{\max\{r_1, q^*\}}}. \end{aligned}$$

To prove the optimality of the exponent $\frac{1}{r_1} - \frac{1}{\max\{r_1, q^*\}}$ we just need to consider $r_1 \leq q^*$; let us consider

$$A_{n_2} : \ell_p \times \ell_q \rightarrow \mathbb{K}$$

given by

$$A_{n_2}(x, y) = x_1 \sum_{j=1}^{n_2} y_j.$$

If

$$\left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} |A_{n_2}(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} \cdot r_2} \right)^{\frac{1}{r_2}} \leq C n_1^{t_1} n_2^{t_2} \|A_{n_2}\|,$$

for certain t_1, t_2 , then

$$n_2^{\frac{1}{r_1}} \leq C n_1^{t_1} n_2^{t_2} n_2^{\frac{1}{q^*}}.$$

Since n_2 is arbitrary, we conclude that

$$t_2 \geq \frac{1}{r_1} - \frac{1}{q^*} = \frac{1}{r_1} - \frac{1}{\max\{r_1, q^*\}}.$$

Now let us prove the optimality of the exponent $\frac{1}{r_2}$ of n_1 when $r_1 \geq q^*$. Consider

$$A_n : \ell_p \times \ell_q \rightarrow \mathbb{K}$$

given by

$$A_n(x, y) = \sum_{j=1}^n x_j y_j.$$

Since $\|A_n\| = 1$, if

$$\left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} |A_n(e_i, e_j)|^{r_1} \right)^{\frac{1}{r_1} r_2} \right)^{\frac{1}{r_2}} \leq C n_1^{t_1} \|A_n\|,$$

for a certain t_1 and all n_1, n_2 , then considering $n_1 = n_2 = n$ we have

$$n^{\frac{1}{r_2}} \leq C n^{t_1}.$$

Since n is arbitrary, we find

$$t_1 \geq \frac{1}{r_2},$$

which concludes the proof. \square

7 Remarks on the multilinear case

In this final Section we comment on the sharp exponent problem for multilinear versions of the Hardy-Littlewood inequality, in the spirit of [26]. Quite recently Dimant and Sevilla-Peris established the existence of a constant $C_{m,p} \geq 1$, such that

$$\left(\sum_{i_1, \dots, i_m=1}^{\infty} |T(e_{i_1}, \dots, e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{m,p} \|T\| \quad (18)$$

for all m -linear operators $T: \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$, with $m \geq 1$ and $p \in (m, 2m)$. Furthermore, they have also shown that the exponent

$$e := \frac{p}{p-m}$$

is sharp in the sense that it cannot be replaced by any $a < \frac{p}{p-m}$ in (18).

As the condition $p < 2m$ is in order, it follows readily that

$$\frac{1}{\frac{p}{p-m}} + \dots + \frac{1}{\frac{p}{p-m}} = \frac{m(p-m)}{p} < \frac{m+1}{2} - \frac{m}{p},$$

which, having in mind the Kahane-Salem-Zygmund inequality, indicates that the optimal exponents $\frac{p}{p-m}$ seem to be sub-optimal in the anisotropic setting. The main result of this section shows, in particular, that in fact the optimal exponents $\frac{p}{p-m}$ are not optimal in the anisotropic stronger sense.

Next definition seems to play a decisive role in the sharp exponent problem for multilinear operators.

Definition 7.1. An m -uple of exponents (q_1, \dots, q_m) for which a Hardy-Littlewood type inequality holds and that for any $\varepsilon_j > 0$ and any $j = 1, \dots, m$, there is no Hardy-Littlewood inequality for the m -uple of exponents

$$(q_1, \dots, q_{j-1}, q_j - \varepsilon_j, q_{j+1}, \dots, q_m)$$

is called globally sharp.

A careful application of the tools and reasoning developed in Section 2, combined with techniques from the theory of absolutely summing operators, [15], yields the following result, that extends the reach of (18) and Theorem 5.2 with globally sharp exponents:

Theorem 7.2. *Let $m \geq 3$ be a positive integer and $p \geq 2m - 2$ be a real number. Then there is a constant $C_{m,p} \geq 1$ such that*

$$\left(\sum_{i_1=1}^n \left(\sum_{i_2, \dots, i_m=1}^n |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2(m-1)p}{mp-2m+2}} \right)^{\frac{mp-2m+2}{2(m-1)p} \times \frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \leq C_{m,p} \|T\|$$

for all m -linear operators $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n . Moreover the m -uple of exponents $(\frac{2p}{p-2}, \frac{2(m-1)p}{mp-2m+2}, m-1 \text{ times}, \frac{2(m-1)p}{mp-2m+2})$ is globally sharp.

Proof. Let $T : \ell_p \times \dots \times \ell_p \rightarrow \mathbb{K}$ be a continuous m -linear form. By the Khinchine inequality, there is a constant $K_{m,p} \geq 1$ such that

$$\begin{aligned} & \left(\sum_{i_2, \dots, i_m=1}^n \left(\sum_{i_1=1}^n |T(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})|^2 \right)^{\frac{1}{2} \times \frac{2(m-1)p}{(m-1)p+p-2(m-1)}} \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)p}} \\ & \leq K_{m,p} \left(\sum_{i_2, \dots, i_m=1}^n \int_0^1 \left| \sum_{i_1=1}^n r_{i_1}(t) T(x_{i_1}^{(1)}, \dots, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)}) \right|^{\frac{2(m-1)p}{(m-1)p+p-2(m-1)}} dt \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)p}} \\ & = K_{m,p} \left(\int_0^1 \sum_{i_2, \dots, i_m=1}^n \left| T \left(\sum_{i_1=1}^n r_{i_1}(t) x_{i_1}^{(1)}, \dots, x_{i_{m-1}}^{(m-1)}, x_{i_m}^{(m)} \right) \right|^{\frac{2(m-1)p}{(m-1)p+p-2(m-1)}} dt \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)p}}. \end{aligned}$$

Combining the previous inequality with Theorem 5.2 for the $(m-1)$ -linear operator $T(\sum_{i_1=1}^n r_{i_1}(t) x_{i_1}^{(1)}, \cdot, \dots, \cdot)$ we conclude that there is a constant $L_{m,p} \geq 1$ such that

$$\begin{aligned} & \left(\sum_{i_2, \dots, i_m=1}^n \left(\sum_{i_1=1}^n |T(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})|^2 \right)^{\frac{1}{2} \times \frac{2(m-1)p}{(m-1)p+p-2(m-1)}} \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)p}} \\ & \leq L_{m,p} \sup_{t \in [0,1]} \left\| T \left(\sum_{i_1=1}^n r_{i_1}(t) x_{i_1}^{(1)}, \cdot, \dots, \cdot \right) \right\| \prod_{j=2}^m \left\| (x_{i_j}^{(j)})_{i_j=1}^n \right\|_{w,p^*} \\ & \leq L_{m,p} \|T\| \sup_{t \in [0,1]} \left\| \sum_{i_1=1}^n r_{i_1}(t) x_{i_1}^{(1)} \right\| \prod_{j=2}^m \left\| (x_{i_j}^{(j)})_{i_j=1}^n \right\|_{w,p^*}. \end{aligned}$$

Since (see [15, page 284])

$$\left\| (x_{i_1}^{(1)})_{i_1=1}^n \right\|_{w,1} = \max \left\| \sum_{i_1=1}^n \varepsilon_{i_1} x_{i_1}^{(1)} : \varepsilon_{i_1} = \pm 1, i = 1, \dots, n \right\|,$$

we have

$$\begin{aligned} & \left(\sum_{i_2, \dots, i_m=1}^n \left(\sum_{i_1=1}^n |T(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})|^2 \right)^{\frac{1}{2} \times \frac{2(m-1)}{(m-1)p+p-2(m-1)}} \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)}} \\ & \leq L_{m,p} \|T\| \left\| (x_{i_1}^{(1)})_{i_1=1}^n \right\|_{w,1} \prod_{j=2}^m \left\| (x_{i_j}^{(j)})_{i_j=1}^n \right\|_{w,p^*}. \end{aligned}$$

Since $p \geq 2m - 2$, by the Minkowski inequality (12) we conclude that

$$\begin{aligned} & \left(\sum_{i_1=1}^n \left(\sum_{i_2, \dots, i_m=1}^n |T(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})|^{\frac{2(m-1)p}{(m-1)p+p-2(m-1)}} \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)p} \times 2} \right)^{\frac{1}{2}} \\ & \leq L_{m,p} \|T\| \left\| (x_{i_1}^{(1)})_{i_1=1}^n \right\|_{w,1} \prod_{j=2}^m \left\| (x_{i_j}^{(j)})_{i_j=1}^n \right\|_{w,p^*}. \end{aligned}$$

Thus, T is multiple $\left(2, \frac{2(m-1)p}{mp-2(m-1)}, \dots, \frac{2(m-1)p}{mp-2(m-1)}; 1, p^*, \dots, p^*\right)$ -summing. By the Anisotropic Regularity Principle we conclude that

$$\begin{aligned} & \left(\sum_{i_1=1}^n \left(\sum_{i_2, \dots, i_m=1}^n |T(x_{i_1}^{(1)}, \dots, x_{i_m}^{(m)})|^{\frac{2(m-1)p}{(m-1)p+p-2(m-1)}} \right)^{\frac{(m-1)p+p-2(m-1)}{2(m-1)p} \times \frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \\ & \leq L_{m,p} \|T\| \prod_{j=1}^m \left\| (x_{i_j}^{(j)})_{i_j=1}^n \right\|_{w,p^*} \end{aligned}$$

and, recalling Theorem 3.2, the proof is done. Since

$$\frac{1}{\frac{2(m-1)p}{(m-1)p+p-2(m-1)}} + \overset{m-1 \text{ times}}{+} \frac{1}{\frac{2(m-1)p}{(m-1)p+p-2(m-1)}} + \frac{1}{\frac{2p}{p-2}} = \frac{m+1}{2} - \frac{m}{p},$$

by the Kahane–Salem–Zygmund we conclude that $\left(\frac{2p}{p-2}, \frac{2(m-1)p}{mp-2m+2}, \overset{m-1 \text{ times}}{\dots}, \frac{2(m-1)p}{mp-2m+2}\right)$ is globally sharp. \square

It is worth to note that Theorem 7.2 provides new m -uples of optimal exponents for multilinear Hardy–Littlewood type inequalities that were not covered by the preceding literature.

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